A strong limit theorem on gambling systems

Article *in* Journal of Multivariate Analysis · February 2003
DOI: 10.1016/S0047-259X(02)00054-4

Jinting Wang
Beijing Jiaotong University

Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE DIRECT
Journal of Multivariate Analysis 84 (2003) 262–273
Journal of Multivariate Analysis
http://www.elsevier.com/locate/jmva

A strong limit theorem on gambling systems

Wen Liu[a,]* and Jinting Wang[b]

[a] Department of Mathematics, Hebei University of Technology, Tianjin 300130, People's Republic of China
[b] Department of Mathematics, Northern Jiaotong University, Beijing 100044, People's Republic of China

Received 10 December 1999

Abstract

A strong limit theorem on gambling system for Bernoulli sequences is extended to the sequences of arbitrary discrete random variables by using the conditional probabilities. Furthermore, by allowing the selection function to take values in an interval, the conception of random selection is generalized. In the proof an approach of applying the differentiation of measure on a net to the investigation of the strong limit theorem is proposed.

AMS 1991 subject classifications: 60F15; 60F99

Keywords: Strong limit theorem; Gambling system; Selection function

1. Introduction

Consider a sequence of Bernoulli trials, and suppose that at each trial the bettor has the free choice of whether or not to bet. A theorem on gambling system asserts that under any non-anticipative system the successive bets form a sequence of Bernoulli trial with unchanged probability for success. The importance of this statement was recognized by von Mises, who introduced the impossibility of a successful gambling system as a fundamental axiom (cf. [1, p. 91; 2, p. 186]). This topic was discussed still further by Kolmogorov [4] and Liu and Wang [5]. The purpose of this paper is to extend the discussion to the case of dependent random variables by using the differentiation of measure on a net (cf. [3, p. 373]). We also

*Corresponding author.
E-mail address: wjting@yahoo.com (J. Wang).

0047-259X/03/$ - see front matter
doi:10.1016/S0047-259X(02)00054-4

extend the notion of random selection, which is the crucial part of the gambling system, by allowing the selection function to take values in an interval.

Let $S = \{t_0, t_1, \dots\}$ be a countable (finite or denumerable) set, $\{X_n, \ n \geqslant 1\}$ a sequence of random variables taking values in S with the joint distribution

$$P(X_1 = x_1, \dots, X_n = x_n) = p(x_1, \dots, x_n) > 0, \quad x_i \in S, \ 1 \leqslant i \leqslant n. \tag{1}$$

In order to extend the idea of random selection (cf. [6, p. 277]), we first give a set of real-valued functions $f_n(x_1, \dots, x_n)$ defined on $S^n (n = 1, 2, \dots)$, which will be called the A-valued selection function if they take values in a set A of real numbers. Then let

$$Y_1 = y_1, \tag{2}$$

$$Y_{n+1} = f_n(X_1, \dots, X_n), \quad n \geqslant 1, \tag{3}$$

where y_1 is an arbitrary real number. Let $\delta_i(j)$ be the Kronecker delta function on S, that is, for $i, j \in S$,

$$\delta_i(j) = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Denote

$$p(x_n|x_1, \dots, x_{n-1}) = P(X_n = x_n | X_1 = x_1, \dots, X_{n-1} = x_{n-1}),$$
$$x_i \in S, \quad 1 \leqslant i \leqslant n, \quad n \geqslant 2. \tag{4}$$

In order to explain the real meaning of the extended notion of random selection, we consider the following gambling model. Let $\{X_n, \ n \geqslant 1\}$ be a sequence of random variables with the joint distribution (1), and g be a real-valued function (not necessarily positive) define on S. Interpret X_n as the result of the nth trial, the type of which may change at each step. Let $\mu_n = Y_n g(X_n)$ denote the gain of the bettor at the nth trial, where Y_n represents the bet size, $g(X_n)$ is determined by the gambling rules, and $\{Y_n, \ n \geqslant 1\}$ is called a gambling system or a random selection system. The bettor's strategy is to determine $Y_n (n \geqslant 2)$ by the results of the first $n-1$ trials. Let the entrance fee that the bettor pays at the nth trial be b_n. Also suppose that b_n depends on $X_1, \dots, X_{n-1}$ as $n \geqslant 2$, and b_1 is a constant. Thus $\sum_{k=1}^n Y_k g(X_k)$ represents the total gain in the first n trials, $\sum_{k=1}^n b_k$ the accumulated entrance fees, and $\sum_{k=1}^n [Y_k g(X_k) - b_k]$ the accumulated net gain. Motivated by the classical definition of " fairness" of game of chance (cf. [2, pp. 233–236]), we introduce the following

Definition. The game is said to be fair, if for almost all $\omega \in \{\omega : \sum_{k=1}^\infty Y_k = \infty\}$ the accumulated net gain in the first n trials is to be of smaller order of magnitude than the accumulated stake $\sum_{k=1}^n Y_k$ as n tends to infinity, that is

$$\lim_{n \to \infty} \frac{1}{\sum_{k=1}^n Y_k} \sum_{k=1}^n [Y_k g(X_k) - b_k] = 0 \quad \text{a.s. on} \quad \left\{ \omega : \sum_{k=1}^\infty Y_k = \infty \right\}. \tag{5}$$

In Section 3 some sufficient conditions for the fairness of game are given as corollaries of the main results of this paper.

2. Main result

Theorem 1. *Let* $\{X_n,\ n \geqslant 1\}$ *be a sequence of random variables on the probability space* $(\Omega, \mathscr{F}, P)$ *with the joint distribution* (1), $Y_n (n = 1, 2, \ldots)$ *defined by* (2) *and* (3), $\{\sigma_n,\ n \geqslant 1\}$ *be a sequence of non-negative random variables on* $(\Omega, \mathscr{F}, P)$, $t_i \in S$, *and* $\alpha > 0$ *be a constant. Let*

$$D = \left\{ \omega : \lim_{n \to \infty} \sigma_n = \infty \right\}, \tag{6}$$

and $D(\alpha, t_i)$ *be the set of sample points* $\omega \in D$ *satisfying the following condition*:

$$\limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=2}^{n} [Y_k^2 e^{\alpha |Y_k|} p(t_i | X_1, \ldots, X_{k-1})] = M(\omega) < \infty. \tag{7}$$

Then

$$\lim_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} Y_k [\delta_{t_i}(X_k) - p(t_i | X_1, \ldots, X_{k-1})] = 0 \quad \text{a.s. on } D(\alpha, t_i). \tag{8}$$

Remark. The example to illustrate condition (7) will be given in Corollary 2, where we assume that $\{Y_n,\ n \geqslant 1\}$ is bounded and σ_n is taken to be $\sum_{k=1}^{n} Y_k$.

Proof. Let

$$D_{x_1 \cdots x_n} = \{\omega : X_k = x_k, 1 \leqslant k \leqslant n\},\ \ x_k \in S,\ \ 1 \leqslant k \leqslant n. \tag{9}$$

Then

$$P(D_{x_1 \cdots x_n}) = p(x_1, \ldots, x_n). \tag{10}$$

$D_{x_1 \cdots x_n}$ is called the nth-order cylinder set. Let $\mathscr{N}_n$ be the collection of nth cylinder sets, $\mathscr{N}$ be the collection consisting of $\emptyset, \Omega$, and all cylinder sets, and λ be a non-zero real number. Define a set function μ on $\mathscr{N}$ as follows: Let

$$\mu(D_{x_1 \cdots x_n}) = \frac{\exp\{\lambda \sum_{k=1}^{n} y_k \delta_{t_i}(x_k)\} p(x_1, \ldots, x_n)}{\prod_{k=2}^{n} [1 + (e^{\lambda y_k} - 1) p(t_i | x_1, \ldots, x_{k-1})]} \tag{11}$$

as $n \geqslant 2$, where y_1 is an arbitrary real number, and

$$y_k = f_{k-1}(x_1, \ldots, x_{k-1}), \quad k \geqslant 2. \tag{12}$$

Also let

$$\mu(D_{x_1}) = \sum_{x_2 \in S} \mu(D_{x_1 x_2}), \tag{13}$$

$$\mu(\Omega) = \sum_{x_1 \in S} \mu(D_{x_1}). \tag{14}$$

Noting that as $n \geqslant 2$,

$$p(x_1, \ldots, x_n) = p(x_1, \ldots, x_{n-1}) p(x_n | x_1, \ldots, x_{n-1}),$$

and y_n depends only on $x_1, \ldots, x_{n-1}$, we have by (10)

$$\begin{aligned}\sum_{x_n \in S} \mu(D_{x_1 \cdots x_n}) &= \mu(D_{x_1 \cdots x_{n-1}}) \sum_{x_n \in S} \frac{\exp\{\lambda y_n \delta_{t_i}(x_n)\} p(x_n | x_1, \ldots, x_{n-1})}{1 + (e^{\lambda y_n} - 1) p(t_i | x_1, \ldots, x_{n-1})} \\ &= \mu(D_{x_1 \cdots x_{n-1}}) \left[\sum_{x_n = t_i} + \sum_{x_n \neq t_i} \right] = \mu(D_{x_1 \cdots x_{n-1}}) \\ &\quad \times \left[\frac{e^{\lambda y_n} p(t_i | x_1, \ldots, x_{n-1})}{1 + (e^{\lambda y_n} - 1) p(t_i | x_1, \ldots, x_{n-1})} \right. \\ &\quad \left. + \frac{1 - p(t_i | x_1, \ldots, x_{n-1})}{1 + (e^{\lambda y_n} - 1) p(t_i | x_1, \ldots, x_{n-1})} \right] \\ &= \mu(D_{x_1 \cdots x_{n-1}}). \end{aligned} \tag{15}$$

It follows from (13)–(15) that μ is a measure on $\mathcal{N}$. Since $\mathcal{N}$ is semifield, μ has an unique extension to the σ-field $\sigma(\mathcal{N})$. Let

$$t_n(\lambda, \omega) = \sum_{D \in \mathcal{N}_n} \frac{\mu(D)}{P(D)} I_D,$$

where I_D denotes the indicator function of D, that is,

$$t_n(\lambda, \omega) = \frac{\mu(D_{X_1(\omega) \cdots X_n(\omega)})}{P(D_{X_1(\omega) \cdots X_n(\omega)})}. \tag{16}$$

It is easy to see that $\{\mathcal{N}_n, \ n \geqslant 1\}$ is a net. By the differentiation on a net (cf. [3, p. 373]), there exists $A(\lambda, t_i) \in \sigma(\mathcal{N}) \subset \mathcal{F}$, $P(A(\lambda, t_i)) = 1$, such that

$$\lim_{n \to \infty} t_n(\lambda, \omega) = a \quad \text{finite number}, \ \omega \in A(\lambda, t_i). \tag{17}$$

This implies that

$$\limsup_{n \to \infty} \frac{1}{\sigma_n} \ln t_n(\lambda, \omega) \leqslant 0, \quad \omega \in A(\lambda, t_i) \cap D. \tag{18}$$

By (3), (9), and (10)–(12), we have when $n \geqslant 2$,

$$\ln t_n(\lambda, \omega) = \lambda \sum_{k=1}^{n} Y_k \delta_{t_i}(X_k) - \sum_{k=2}^{n} \ln[1 + (e^{\lambda Y_k} - 1) p(t_i | X_1, \ldots, X_{k-1})]. \tag{19}$$

It follows from (18) and (19) that

$$\limsup_{n \to \infty} \frac{1}{\sigma_n} \left\{ \lambda \sum_{k=1}^{n} Y_k \delta_{t_i}(X_k) - \sum_{k=2}^{n} \ln[1 + (e^{\lambda Y_k} - 1) p(t_i | X_1, \ldots, X_{k-1})] \right\} \leqslant 0, \quad \omega \in A(\lambda, t_i) \cap D. \tag{20}$$

In virtue of the property of superior limit:

$$\limsup_{n \to \infty} (a_n - b_n) \leqslant 0 \Rightarrow \limsup_{n \to \infty} (a_n - c_n) \leqslant \limsup_{n \to \infty} (b_n - c_n), \tag{21}$$

from (20) we have

$$\limsup_{n\to\infty} \frac{1}{\sigma_n}\left[\sum_{k=1}^{n} \lambda Y_k \delta_{t_i}(X_k) - \sum_{k=2}^{n} \lambda Y_k p(t_i|X_1, \ldots, X_{k-1})\right]$$
$$\leqslant \limsup_{n\to\infty} \frac{1}{\sigma_n}\left\{\sum_{k=2}^{n} \ln[1+(e^{\lambda Y_k}-1)p(t_i|X_1,\ldots,X_{k-1})] - \sum_{k=2}^{n} \lambda Y_k p(t_i|X_1,\ldots,X_{k-1})\right\},$$
$$\omega \in A(\lambda, t_i)\cap D. \tag{22}$$

Let $0<\lambda<\alpha$. Dividing the two sides of (22) by λ, and using the inequalities

$$\ln(1+x)\leqslant x \ (x>-1), \quad 0\leqslant e^x - 1 - x \leqslant x^2 e^{|x|}. \tag{23}$$

Combining (22) and (7), we have

$$\limsup_{n\to\infty} \frac{1}{\sigma_n}\left[\sum_{k=1}^{n} Y_k \delta_{t_i}(X_k) - \sum_{k=2}^{n} Y_k p(t_i|X_1, \ldots, X_{k-1})\right]$$
$$\leqslant \limsup_{n\to\infty} \frac{1}{\sigma_n}\sum_{k=2}^{n}\left[\frac{1}{\lambda}(e^{\lambda Y_k}-1-\lambda Y_k)p(t_i|X_1,\ldots,X_{k-1})\right]$$
$$\leqslant \limsup_{n\to\infty} \frac{1}{\sigma_n}\sum_{k=2}^{n} [\lambda Y_k^2 e^{\alpha|Y_k|}p(t_i|X_1,\ldots,X_{k-1})] = \lambda M(\omega),$$
$$\omega\in D(\alpha,t_i)\cap A(\lambda,t_i). \tag{24}$$

Assume $0<\lambda_k<\alpha$ $(k=1,2,\ldots)$, $\lambda_k\to 0$ (as $k\to\infty$), and let $A^*(t_i)=\bigcap_{k=1}^{\infty} A(\lambda_k,t_i)$. Then by (24) we have for all $k\geqslant 1$,

$$\limsup_{n\to\infty} \frac{1}{\sigma_n}\left[\sum_{k=1}^{n} Y_k \delta_{t_i}(X_k) - \sum_{k=2}^{n} Y_k p(t_i|X_1, \ldots, X_{k-1})\right] \leqslant \lambda_k M(\omega),$$
$$\omega\in D(\alpha,t_i)\cap A(\lambda_j,t_i). \tag{25}$$

Since $\lambda_k\to 0$, it follows from (25) that

$$\limsup_{n\to\infty} \frac{1}{\sigma_n}\left[\sum_{k=1}^{n} Y_k \delta_{t_i}(X_k) - \sum_{k=2}^{n} Y_k p(t_i|X_1, \ldots, X_{k-1})\right] \leqslant 0,$$
$$\omega\in D(\alpha,t_i)\cap A^*(t_i). \tag{26}$$

Assume $-\alpha<\tau_k<0 (k=1,2,\ldots), \tau_k\to 0$ (as $k\to\infty$), and let $A_*(t_i)=\bigcap_{k=1}^{\infty} A(\tau_k,t_i)$. Imitating the deduction of (26), we have by (22),

$$\liminf_{n\to\infty} \frac{1}{\sigma_n}\left[\sum_{k=1}^{n} Y_k \delta_{t_i}(X_k) - \sum_{k=2}^{n} Y_k p(t_i|X_1, \ldots, X_{k-1})\right] \geqslant 0,$$
$$\omega\in D(\alpha,t_i)\cap A_*(t_i). \tag{27}$$

Letting $A(t_i) = A^*(t_i) \cap A_*(t_i)$, from (26) and (27) we have

$$\lim_{n\to\infty} \frac{1}{\sigma_n}\left[\sum_{k=1}^{n} Y_k\delta_{t_i}(X_k) - \sum_{k=2}^{n} Y_k p(t_i|X_1, \dots, X_{k-1})\right] = 0,$$
$$\omega \in D(\alpha, t_i) \cap A(t_i). \tag{28}$$

Since $P(A(t_i)) = 1$, (8) follows from (28) directly. This completes the proof of the theorem. □

Theorem 2. *Let* $S = \{t_1, \dots, t_N\}$. *Then*

$$\lim_{n\to\infty} \frac{1}{\sigma_n}\sum_{k=2}^{n} Y_k\{g(X_k) - E[g(X_k)|X_1, \dots, X_{k-1}]\} = 0 \quad a.s.\ on\ D(\alpha), \tag{29}$$

where $D(\alpha) = \bigcap_{i=1}^{N} D(\alpha, t_i)$.

Proof. It is easy to see that

$$g(X_k) = \sum_{i=1}^{N} g(t_i)\delta_{t_i}(X_k), \tag{30}$$

$$E[\delta_{t_i}(X_k)|X_1, \dots, X_{k-1}] = p(t_i|X_1, \dots, X_{k-1}), \quad k \geqslant 2. \tag{31}$$

We have by (8), (30) and (31),

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\sigma_n}\sum_{k=2}^{n} Y_k\{g(X_k) - E[g(X_k)|X_1, \dots, X_{k-1}]\} \\
&\quad = \lim_{n\to\infty} \frac{1}{\sigma_n}\sum_{k=2}^{n}\sum_{i=1}^{N} Y_k g(t_i)[\delta_{t_i}(X_k) - p(t_i|X_1, \dots, X_{k-1})] \\
&\quad = \sum_{i=1}^{N} g(t_i)\lim_{n\to\infty} \frac{1}{\sigma_n}\sum_{k=2}^{n} Y_k[\delta_{t_i}(X_k) - p(t_i|X_1, \dots, X_{k-1})] \\
&\quad = 0 \quad \text{a.s. on } D(\alpha),
\end{aligned} \tag{32}$$

that is, (29) holds. □

3. Some corollaries

In Corollaries 1–5 we consider a simple gambling model, in which g is taken to be $\delta_{t_i}(j) (j \in S)$ for fixed t_i.

Corollary 1. *Let*

$$S_n(t_i,\omega) = \sum_{k=1}^{n} Y_k \delta_{t_i}(X_k), \tag{33}$$

$$D_1(\alpha, t_i) = \left\{ \omega : \sum_{k=2}^{\infty} Y_k^2 e^{\alpha|Y_k|} p(t_i|X_1, \dots, X_{k-1}) = \infty \right\}, \tag{34}$$

where α *is an arbitrary positive constant. Then*

$$\lim_{n\to\infty} \frac{S_n(t_i,\omega) - \sum_{k=2}^{n} Y_k p(t_i|X_1,\dots,X_{k-1})}{\sum_{k=2}^{n} Y_k^2 e^{\alpha|Y_k|} p(t_i|X_1,\dots,X_{k-1})} = 0 \quad a.s. \text{ on } D_1(\alpha, t_i). \tag{35}$$

Proof. Let

$$\sigma_n = \sum_{k=2}^{n} Y_k^2 e^{\alpha|Y_k|} p(t_i|X_1,\dots,X_{k-1}), \quad n \geqslant 2. \tag{36}$$

Then

$$\mu(\omega) = 1, \quad \omega \in D_1(\alpha, t_1) = D.$$

Hence

$$D(\alpha, t_i) = D_1(\alpha, t_i). \tag{37}$$

Eq. (33) follows from (8) and (37) directly. □

Corollary 2. *Assume that the selection functions* $f_n(x_1,\dots,x_n)(n = 1,2,\dots)$ *take values in an interval* $[0,b]$ (*that is,* $f_n(x_1,\dots,x_n)$ *is* $[0,b]$*-valued selection function*), *and let*

$$D_2 = \left\{ \omega : \sum_{k=1}^{\infty} Y_k = \infty \right\}. \tag{38}$$

Then

$$\lim_{n\to\infty} \frac{1}{\sum_{k=1}^{n} Y_k} \left[S_n(t_i,\omega) - \sum_{k=2}^{n} Y_k p(t_i|X_1,\dots,X_{k-1}) \right] = 0 \quad a.s. \text{ on } D_2. \tag{39}$$

Proof. Let

$$\sigma_n = \sum_{k=1}^{n} Y_k, \tag{40}$$

and let $0 \leqslant y_1 \leqslant b$. Then we have

$$0 \leqslant Y_k \leqslant b, \quad k \geqslant 1. \tag{41}$$

It follows from (40) and (41) that

$$\frac{1}{\sigma_n}\sum_{k=2}^{n} Y_k^2 e^{\alpha Y_k} p(t_i|X_1, \dots, X_{k-1}) \leqslant \frac{\sum_{k=2}^{n} Y_k^2 e^{\alpha Y_k}}{\sum_{k=1}^{n} Y_k} \leqslant b e^{\alpha b} < \infty. \tag{42}$$

Eqs. (40) and (42) imply that

$$D(\alpha, t_i) = D = D_2. \tag{43}$$

Eq. (39) follows from (8) and (43) directly. □

Remark. In the above gambling model $\mu_n = Y_n$ if the event $\{X_n = t_i\}$ occurs and $\mu_n = 0$ if $\{X_n = t_i\}$ does not occur. Hence $S_n(t_i, \omega)$ and $\sum_{k=1}^{n} Y_k$ represent, respectively, the total gain and the total amount winnable of the bettor at the first n trials, and where "fairness" is defined to mean that the ratio of the net gain to the total amount winnable goes to 0 as the number of bets goes to infinity. By (39) and (5) we assert that if the entrance fee b_k that the bettor pays at the kth trial is taken to be $Y_k p(t_i|X_1, \dots, X_{k-1})(k \geqslant 2)$ then the game is fair, that is, no strategy can increase the bettor's return on investment.

Corollary 3. *Under the assumption of Corollary* 2, *we have*

$$\lim_{n\to\infty} \frac{S_n(t_i, \omega)}{\sum_{k=2}^{n} Y_k p(t_i|X_1, \dots, X_{k-1})} = 1 \quad \text{a.s. on } D_3(t_i), \tag{44}$$

where

$$D_3(t_i) = \left\{ \omega : \sum_{k=2}^{\infty} Y_k p(t_i|X_1, \dots, X_{k-1}) = \infty \right\}. \tag{45}$$

Proof. Let

$$\sigma_n = \sum_{k=2}^{n} Y_k p(t_i|X_1, \dots, X_{k-1}), \quad n \geqslant 2. \tag{46}$$

Then

$$\begin{aligned} &\frac{1}{\sigma_n}\sum_{k=2}^{n} Y_k^2 e^{\alpha Y_k} p(t_i|X_1, \dots, X_{k-1}) \\ &\quad \leqslant \frac{b}{\sum_{k=2}^{n} Y_k p(t_i|X_1, \dots, X_{k-1})} \sum_{k=2}^{n} Y_k e^{\alpha b} p(t_i|X_1, \dots, X_{k-1}) \\ &\quad = b e^{\alpha b} < \infty. \end{aligned} \tag{47}$$

Eqs. (46) and (47) imply that

$$D(\alpha, t_i) = D = D_3(t_i). \tag{48}$$

It follows from (48) and (8) that

$$\lim_{n\to\infty} \frac{1}{\sum_{k=2}^{n} Y_k p(t_i|X_1, \ldots, X_{k-1})} \sum_{k=1}^{n} Y_k[\delta_{t_i}(X_k) - p(t_i|X_1, \ldots, X_{k-1})] = 0$$

a.s. on $D_3(t_i)$.

This implies (44) obviously. □

Corollary 4. *Under the assumption of Corollary* 3, *we have*

$$D_3(t_i) = D_4(t_i) \quad a.s., \tag{49}$$

where

$$D_4(t_i) = \left\{\omega : \sum_{k=1}^{\infty} Y_k \delta_{t_i}(X_k) = \infty\right\}. \tag{50}$$

Proof. Corollary 3 implies

$$D_3(t_i) \subset D_4(t_i) \quad \text{a.s.} \tag{51}$$

Now we come to the proof of the opposite inclusion. Let $\omega_0 \in A(1, t_i) \cap D_3'(t_i)$, where $D_3'(t_i)$ denotes the complement of $D_3(t_i)$. We have by (17) and (19),

$$\begin{aligned}\lim_{n\to\infty} t_i(1, \omega_0) &= \lim_{n\to\infty} \frac{\exp\{\sum_{k=1}^{n} Y_k(\omega_0)\delta_{t_i}(X_k(\omega_0))\}}{\prod_{k=2}^{n}[1 + (e^{Y_k(\omega_0)} - 1)p(t_i|X_1(\omega_0), \ldots, X_{k-1}(\omega_0))]} \\ &= \text{finite number}.\end{aligned} \tag{52}$$

By using the inequality $e^x - 1 \leqslant xe^x$, we have

$$\begin{aligned}0 &\leqslant \sum_{k=2}^{\infty} (e^{Y_k(\omega_0)} - 1)p(t_i|X_1(\omega_0), \ldots, X_{k-1}(\omega_0)) \\ &\leqslant \sum_{k=2}^{\infty} Y_k(\omega_0)e^b p(t_i|X_1(\omega_0), \ldots, X_{k-1}(\omega_0)) < \infty.\end{aligned} \tag{53}$$

By (53) and the convergence theorem of infinite product we have

$$0 < \prod_{k=2}^{\infty} [1 + (e^{Y_k(\omega_0)} - 1)p(t_i|X_1(\omega_0), \ldots, X_{k-1}(\omega_0))] < \infty. \tag{54}$$

Eqs. (52) and (54) imply

$$\sum_{k=1}^{\infty} Y_k(\omega_0)\delta_{t_i}(X_k(\omega_0)) < \infty, \tag{55}$$

that is, $\omega_0 \in D_4'(t_i)$. Hence $A(1,t_i)\cap D_3'(t_i)\subset A(1,t_i)\cap D_4'(t_i)$, that is,

$$A(1,t_i)\cap D_4(t_i)\subset A(1,t_i)\cap D_3(t_i). \tag{56}$$

Since $P(A(1,t_i))=1$, (56) together with (51) implies (49). □

The next corollary is the classical result on impossibility of a successful gambling system.

Corollary 5. *Let $S=\{0,1\}, \{X_n,\ n\geqslant 1\}$ be a sequence of independent and identically distributed random variables with Bernoulli distribution $(1-p,p)$, $\{Y_n,\ n\geqslant 1\}$ is bounded, and*

$$\sum_{k=1}^{\infty} Y_k=\infty \quad a.s.$$

Then

$$\lim_{n\to\infty}\frac{1}{\sum_{k=1}^n Y_k}\sum_{k=1}^n Y_kX_k=p \quad a.s.$$

Proof. Taking $t_i=1$ and noticing that $p(t_i|X_1,\dots,X_{k-1})=p$ in this case, Corollary 5 follows from Corollary 3 directly. □

In the following corollary we consider the general gambling model with $S=\{t_1,\dots,t_N\}$ and the payoff in the nth trial taken to be $Y_kg(X_n)$, where g is an arbitrary function defined on S.

Corollary 6. *Let $S=\{t_1,\dots,t_N\}$. Suppose that $0\leqslant f_n(x_1,\dots,x_n)\leqslant b$ for all $n\geqslant 1$. Then we have*

$$\lim_{n\to\infty}\frac{1}{\sum_{k=1}^n Y_k}\sum_{k=1}^n Y_k\{g(X_k)-E[g(X_k)|X_1,\dots,X_{k-1}]\}=0 \quad a.s.\ on\ D_2. \tag{57}$$

Proof. By (43) we have $D(\alpha)=D_2$. Hence (57) follows from (29) directly. □

Remark. By (57) and (5) we assert that if the entrance fee b_k that the bettor pays at the kth trial is taken to be $Y_kE[g(X_k)|X_1,\dots,X_{k-1}](k\geqslant 2)$, then the game is fair.

4. Some examples

Example 1. Let $\{X_n,\ n\geqslant 1\}$ be a non-homogeneous Markov chain with the state space $S=\{1,2,\dots,N\}$ and the transition matrix

$$P_n=(P_n(j|i)), \qquad P_n(j|i)>0, \quad i,j\in S,\ n\geqslant 2,$$

where $P_n(j|i) = P(X_n = j|X_{n-1} = i)(n \geqslant 2)$. Suppose the random selection system is bounded, that is, $0 \leqslant f_n(x_1, \dots, x_n) \leqslant b < \infty$ for all $n \geqslant 1$. Then

$$\lim_{n\to\infty} \frac{1}{\sum_{k=1}^{n} Y_k} \sum_{k=2}^{n} Y_k\{X_k - E[X_k|X_{k-1}]\} = 0 \quad \text{a.s. on } D_2, \tag{58}$$

and for fixed $i \in S$,

$$\lim_{n\to\infty} \frac{1}{\sum_{k=1}^{n} Y_k} \sum_{k=2}^{n} Y_k[\delta_i(X_k) - P_k(i|X_{k-1})] = 0 \quad \text{a.s. on } D_2. \tag{59}$$

Proof. Taking $g(j) = j(j \in S)$ and noticing that $E[X_k|X_1, \dots, X_{k-1}] = E[X_k|X_{k-1}]$, (58) follows from (57) directly. Taking $g(j) = \delta_i(j)(j \in S)$ and noticing that

$$E[\delta_i(X_k)|X_{k-1}] = P_k(i|X_{k-1}),$$

Eq. (59) also follows from (57) directly. □

Example 2. Let $\{X_n,\ n \geqslant 1\}$ be a homogeneous Markov chain with the state space $S = \{1, 2\}$ and the transition matrix

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \qquad p_{ij} > 0,\ p_{21} \geqslant p_{11},$$

and b be a positive number. Suppose that by the gambling rules the gain of the bettor at the $(n+1)$ th trial be $Y_{n+1}\delta_1(X_{n+1})$. Since $p_{21} \geqslant p_{11}$ means that the conditional probability $P(X_{n+1}|X_n = 2)$ is greater than the conditional probability $P(X_{n+1}|X_n = 1)$, it seems at first that if the bettor employs the following strategy: when $X_n = 2$, let $Y_{n+1} = b$; when $X_n = 1$, skip the $(n+1)$ th bet, that is, let

$$f_n(x_1, \dots, x_n) = \begin{cases} 0 & \text{when } x_n = 1, \\ 1 & \text{when } x_n = 2, \end{cases}$$

he can perhaps gain some advantage, but the Corollary 2 says that he cannot. In fact, we have by this corollary,

$$\lim_{n\to\infty} \frac{1}{\sum_{k=1}^{n} Y_k} \left[S_n(1, \omega) - \sum_{k=2}^{n} Y_k p(1|X_1, \dots, X_{k-1}) \right] = 0 \quad \text{a.s. on } D_2$$

for any bounded gambling system $\{Y_n,\ n \geqslant 1\}$. Hence the above strategy cannot increase the bettor's return on investment.

Acknowledgments

The authors are thankful to the referees for their valuable suggestions.

References

[1] P. Billingsley, Probability and Measure, Wiley, New York, 1986.

[2] W. Feller, An Introduction to Probability Theory and Its Applications, Vol. 1, 2nd Edition, Wiley, New York, 1957.

[3] E. Hewitt, K.R. Stromberg, Real and Abstract Analysis—A Modern Treatment of the Theory of Functions of a Real Variable, Springer, New York, 1978.

[4] A.N. Kolmogorov, On the Logical Foundation of Probability Theory, Lecture Notes in Mathematics, Vol. 1021, Springer, New York, 1982, pp. 1–5.

[5] W. Liu, Z. Wang, A extension of a theorem on gambling systems, J. Multivariate Anal. 55 (1) (1995) 125–132.

[6] M. Loeve, Probability Theory 1, 4th Edition, Springer, New York, 1977.